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N–EXTENDED LOCAL SUPERSYMMETRY OF MASSLESS PARTICLES IN SPACES OF CONSTANT CURVATURE [a]

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We review the unified description of massless spinning particles, living in spaces of constant curvature, in the framework of the pseudoclassical approach with a gauged N-extended worldline supersymmetry and a local $O(N)$ invariance.

In the pseudoclassical approach [1], the spin degrees of freedom of point particles are realized by anticommuting variables which turn into a set of generalized γ-matrices at the quantum level. This approach is essentially supersymmetric, since the consistent treatment of a particle with spin requires twice as many of local worldline supersymmetries as the value of spin.

The mechanics action with a gauged N-extended supersymmetry for a massless particle in Minkowski space was suggested some years ago by Gershun and Tkach [2] and investigated in detail by Howe *et al.* [3]. In particular, it was argued that worldline supersymmetry is compartible with arbitrary gravitational background only for $N \leq 2$. This bound is very natural because of the known problems with formulating the higher-spin dynamics in curved space. Such problems do not in general arise when the background geometry is chosen to be maximally symmetric, although it was believed [3] for a time that Minkowski space is the only background compartible with worldline supersymmetry for $N > 2$. In a recent paper [4] we have extended the Gershun-Tkach (GT) model [2] to the cases of de Sitter (dS) and anti-de Sitter (AdS) spaces. Our construction provides a unified treatment of the dynamics of massless particles in spaces of constant curvature and is based on a hidden conformal invariance.

Howe *et al* [3] demonstrated that in $d = 3 + 1$ dimensions the wave functions in the GT model satisfy the conformally covariant equation for a pure

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spin-$\frac{1}{2}N$ field strength (helicities $\pm\frac{1}{2}N$) [5]. That might apparently have implied conformal invariance of the model for all d and N. This proposal has been proved by Siegel [6] who found the ansatz to obtain the GT model from an explicitly conformal ($O(d,2)$ invariant) mechanics action in d space and 2 time dimensions (Siegel extended, to the higher-spin case, the construction originally used by Marnelius [7] to represent the actions for massless spin-0 and spin-$\frac{1}{2}$ particles in a manifestly conformal form). It turns out [4] that the same $(d+2)$-dimensional action can be used to derive the point particle models with N-extended worldline supersymmetry in the dS and AdS spaces.

We consider the mechanics system in d space and 2 time dimensions with the action [4,6] $S = \int \mathrm{d}\tau \mathcal{L}$ given by

$$\mathcal{L} = \frac{1}{2}\dot{Z}^{\mathcal{A}}\dot{Z}_{\mathcal{A}} + \frac{\mathrm{i}}{2}\Gamma_i{}^{\mathcal{A}}\dot{\Gamma}_{i\mathcal{A}} - \frac{\mathrm{i}}{2}\varphi_{ij}\Gamma_i{}^{\mathcal{A}}\Gamma_{i\mathcal{A}}\ . \tag{1}$$

Here $\varphi_{ij}(\tau)$, $\varphi_{ij} = -\varphi_{ji}$, are Lagrange multipliers, the bosonic $Z^{\mathcal{A}}(\tau)$, $\mathcal{A} = d+1, 0, 1, \ldots, d$, and fermionic $\Gamma_i{}^{\mathcal{A}}(\tau)$, $i = 1, \ldots, N$, dynamical variables are subject to the constraints

$$\eta_{\mathcal{AB}}Z^{\mathcal{A}}Z^{\mathcal{B}} = 0, \qquad Z \neq 0 \tag{2}$$

$$\eta_{\mathcal{AB}}Z^{\mathcal{A}}\Gamma_i{}^{\mathcal{B}} = 0 \tag{3}$$

with $\eta_{\mathcal{AB}} = \mathrm{diag}(- - + \ldots +)$. Hence the variables $Z^{\mathcal{A}}$ parametrize the cone Q in $\mathbf{R}^{d,2}$, whilst $\Gamma_i{}^{\mathcal{A}}$ form n tangent vectors to point Z of the cone.

Along with the explicit global $O(d,2)$ invariance (conformal invariance), the model possesses a rich gauge structure. The action remains unchanged under worldline reparametrizations and local $O(N)$ transformations [4,6]. Moreover, the action is invariant under local N-extended supersymmetry transformations of rather unusual form [4]. These transformations involve an *external* $(d+2)$-vector $W^{\mathcal{A}}$, chosen to satisfy the only requirement $(Z,W) = Z^{\mathcal{A}}W_{\mathcal{A}} \neq 0$ for the worldline $\{Z^{\mathcal{A}}(\tau), \Gamma_i{}^{\mathcal{A}}(\tau), \varphi_{ij}(\tau)\}$ in field, and read as follows

$$\begin{aligned} \delta\Gamma_i{}^{\mathcal{A}} &= Z^{\mathcal{A}}\overset{\bullet}{\alpha}_i - \dot{Z}^{\mathcal{A}}\alpha_i + \frac{\mathrm{i}}{(Z,W)}\Gamma_i{}^{\mathcal{B}}\Gamma_{j\mathcal{B}}\alpha_j W^{\mathcal{A}}, \\ \delta Z^{\mathcal{A}} &= \mathrm{i}\alpha_i\Gamma_i{}^{\mathcal{A}}, \qquad \delta\varphi_{ij} = -\frac{\mathrm{i}}{(Z,W)}\alpha_{[i}\overset{\bullet}{\Gamma}_{j]}{}^{\mathcal{A}}W_{\mathcal{A}}\ . \end{aligned} \tag{4}$$

Here $\overset{\bullet}{\alpha}_i$ denotes an $O(N)$-covariant derivative, $\overset{\bullet}{\alpha}_i = \dot{\alpha}_i - \varphi_{ij}\alpha_j$, and similarly for $\overset{\bullet}{\Gamma}_i{}^{\mathcal{A}}$. The origion of the last term in $\delta\Gamma$ is to preserve the constraint (3).

The expressions (4) become W-independent only on the mass shell. Off-shell, however, the supersymmetry transformations do not commute with the

conformal ones, in spite of the manifest $O(d,2)$ invariance of $\mathcal{L}$! What is the physical origion of the presence of W-terms in (4)? It turns out that the fixing of W breaks the $O(d,2)$-invariance and uniquely specifies some d-dimensional spacetime which is embedded into the compact projective space PQ related to the cone (2). PQ is defined as the set of straight lines through the origion of the cone. Associated to a non-zero $(d+2)$-vector W is the d-dimensional open submanifold $\mathcal{M}_W$ in PQ

$$\mathcal{M}_W = \{\bar{Z}^{\mathcal{A}} \in PQ, \quad e^{-1} \equiv (Z,W)^2 > 0\} \tag{5}$$

which can be parametrized by constrained $d+2$ projective variables of the form

$$\zeta^{\mathcal{A}} = \frac{Z^{\mathcal{A}}}{(Z,W)}, \qquad \zeta^2 = 0, \qquad (Z,W) = 1. \tag{6}$$

Introducing on $\mathcal{M}_W$ the metric $\mathrm{d}s^2 = \mathrm{d}\zeta^{\mathcal{A}}\mathrm{d}\zeta_{\mathcal{A}} = e\mathrm{d}Z^{\mathcal{A}}\mathrm{d}Z_{\mathcal{A}}$, $\mathcal{M}_W$ turns into a spacetime of constant curvature. Three inequivalent choices for W:

$$W^{\mathcal{A}}_{(M)} = (-\frac{1}{\sqrt{2}}, 0, \ldots, 0, \frac{1}{\sqrt{2}}), \;\; W^{\mathcal{A}}_{(AdS)} = (0, \ldots, 0, \frac{1}{r}), \;\; W^{\mathcal{A}}_{(dS)} = (\frac{1}{r}, 0, \ldots, 0)$$

leads to Minkowski, de Sitter (dS) and anti-de Sitter (AdS) spacetimes, respectively; $(\pm 12 r^{-2})$ being the curvature of the dS (AdS) space. The stability group of $W^{\mathcal{A}}$ in $O(d,2)$ is seen to be the symmetry group of the corresponding spacetime. With respect to the symmetry group, ${\Gamma_i}^{\mathcal{A}}$ is naturally decomposed as follows

$$\lambda_i = e(\Gamma_i, W), \qquad {\Psi_i}^{\mathcal{A}} = {\Gamma_i}^{\mathcal{A}} - (\Gamma_i, W)\frac{Z^{\mathcal{A}}}{(Z,W)} \; . \tag{7}$$

Eqs. (5–7) define the reduction of the conformal model (1) to d spacetime dimensions. The variables e and λ_i proves to enter the final Lagrangian as the einbein and N-extended worldline gravitino respectively.

As an illustration, let us apply the reduction procedure described to the case of the AdS space. This space can be parametrized by $d+1$ constrained variables $y^A \equiv \zeta^A$, where $A = d+1, 0, 1, \ldots, d-1$ (note $\zeta^d = r$). For fermionic variables one gets

$$\lambda_i = \frac{1}{r} e {\Gamma_i}^d, \qquad {\Psi_i}^A = {\Gamma_i}^A - \frac{1}{r} y^A {\Gamma_i}^d, \qquad {\Psi_i}^d = 0. \tag{8}$$

The bosonic y^A and fermionic ${\Psi_i}^A$ degrees of freedom are constrained by

$$y^A y_A = -r^2, \qquad y^A \Psi_{iA} = 0 \; . \tag{9}$$

Thus Ψ_i present themselves N tangent vectors to point y of the AdS hyperboloid. Now, the Lagrangian turns into

$$\mathcal{L}_{AdS} = \frac{1}{2e}(\dot{y}^A - \mathrm{i}\lambda_i {\Psi_i}^A)(\dot{y}_A - \mathrm{i}\lambda_j \Psi_{jA}) + \frac{\mathrm{i}}{2}{\Psi_i}^A(\dot{\Psi}_{iA} - f_{ij}\Psi_{jA})\ . \qquad (10)$$

where we have redefined $\varphi_{ij} = f_{ij} + \frac{\mathrm{i}}{e}\lambda_i\lambda_j$. The supersymmetry transformation (4) takes the form

$$\begin{aligned} \delta y^A &= \mathrm{i}\alpha_i {\Psi_i}^A, \qquad \delta {\Psi_i}^A = -\frac{1}{e}\alpha_i(\dot{y}^A - \mathrm{i}\lambda_j {\Psi_j}^A) - \frac{\mathrm{i}}{r^2} y^A {\Psi_i}^B \Psi_{jB}\alpha_j, \\ \delta e &= 2\mathrm{i}\lambda_i\alpha_i, \qquad \delta\lambda_i = \dot{\alpha_i} - f_{ij}\alpha_j, \qquad \delta f_{ij} = -\frac{\mathrm{i}}{r^2}\alpha_{[i}\Psi_{j]A}\dot{y}^A\ . \end{aligned} \qquad (11)$$

It is of interest to reformulate the model in terms of internal (unconstrained) coordinates x^m, $m = 0, 1, \ldots, d-1$, on the AdS space. Then $\mathcal{L}_{AdS}$ takes the form [4]

$$\begin{aligned} \mathcal{L}_{AdS} &= \frac{1}{2e} g_{mn}(\dot{x}^m - \mathrm{i}\lambda_i {\psi_i}^a {e_a}^m)(\dot{x}^n - \mathrm{i}\lambda_j {\psi_j}^b {e_b}^n) \\ &+ \frac{\mathrm{i}}{2}{\psi_i}^a(\dot{\psi}_{ia} - f_{ij}\psi_{ja} + \dot{x}^m {\omega_{ma}}^b \psi_{ib})\ . \end{aligned} \qquad (12)$$

Here g_{mn} is the metric of the AdS space, ${e_m}^a$ and $\omega_{mab} = -\omega_{mba}$ its vielbein and torsion-free spin connection, respectively; a, b are tangent-space indices, $a, b = 0, 1, \ldots, d-1$. The unconstrained fermionic variables ${\psi_i}^a$ carry a tangent-space vector index and are defined by the rule $\psi_{ia} = {e_a}^m \frac{\partial y^B}{\partial x^m}\Psi_{iB}$. Remarkably, $\mathcal{L}_{AdS}$ presents itself a minimal covariantization of the flat-space Lagrangian [2]. The supersymmetry transformations inevitably involve, however, curvature-dependent terms [4].

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